MAI 1 – řešení 10. domácího úkolu

1. „Jednoduché" příklady:

a) $\int (3x-6)^6 dx = \dfrac{(3x-6)^7}{7} + c, \quad x \in R$ $\qquad \left(\int x^6 dx = \dfrac{x^7}{7} + c, \; x \in R \right)$

lze užít „vzorečků" (skorotabulk"):

je-li $\int f(x)dx = F(x) + c$, $x \in (a,b)$, pak

$$\int f(\alpha x + \beta) dx = \frac{F(\alpha x + b)}{\alpha} + c, \quad \alpha \neq 0$$

(v odpovídajícím intervalu x)

a podobně i:

- $\int \sqrt[3]{(1-2x)^2}\, dx = \int (1-2x)^{\frac{2}{3}} dx = \dfrac{(1-2x)^{\frac{5}{3}}}{\frac{5}{3}\cdot(-2)} + c = \dfrac{3}{5}\, \dfrac{(1-2x)^{\frac{5}{3}}}{-2} + c$
  $x \in R$

  $\left(\int x^{\frac{2}{3}} dx = x^{\frac{5}{3}} \cdot \dfrac{3}{5} + c \right)$

- $\int \dfrac{1}{1-2x} dx = -\dfrac{1}{2} \ln|2x-1| + c, \quad x \in (-\infty, \tfrac{1}{2}),\; x \in (\tfrac{1}{2}, +\infty)$

  $\left(\int \dfrac{1}{x} dx = \ln|x| + c, \; x \in (0,+\infty) \vee x \in (-\infty, 0) \right)$

- $\int \dfrac{1}{(3x+4)^4} dx = \int (3x+4)^{-4} dx = \dfrac{(3x-4)^{-3}}{-3 \cdot 3} + c = -\dfrac{1}{9} \cdot \dfrac{1}{(3x-4)^3} + c$
  $x \neq -\dfrac{4}{3}$

- $\int \dfrac{(1-x)^2}{x\sqrt{x}} dx = \int \dfrac{x^2 - 2x + 1}{x\sqrt{x}} dx = \int \left(\sqrt{x} - \dfrac{2}{\sqrt{x}} + \dfrac{1}{(\sqrt{x})^3} \right) dx =$
  $x \in (0, +\infty)$ $\Big| = \dfrac{2}{3} x^{\frac{3}{2}} - 4\sqrt{x} - 2 \cdot \dfrac{1}{\sqrt{x}} + c$

b) • $\displaystyle \int \frac{1}{x^2+4x+8}\,dx = \int \frac{1}{(x+2)^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx$

$\displaystyle = \frac{1}{4}\;\frac{\operatorname{arctg}\left(\frac{x}{2}+1\right)}{\frac{1}{2}} + C = \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}+1\right) + C, \quad x \in \mathbb{R}$

• $\displaystyle \int \frac{x^4}{x^2+1}\,dx = \int \frac{x^4-1+1}{x^2+1}\,dx = \int \frac{(x^2+1)(x^2-1)+1}{x^2+1}\,dx$

(zde je vhodné vydělit $x^4 : (x^2+1)$) $\displaystyle \Big| = \int \left(x^2-1+\frac{1}{x^2+1}\right)dx = \frac{x^3}{3} - x + \operatorname{arctg} x + C,$

$x \in \mathbb{R}$

(dělení: (školní)) $x^4 : (x^2+1) = x^2 - 1$, tj: $\displaystyle \frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$)

$$\begin{array}{l} -(x^4+x^2) \\ \hline \quad -x^2 \\ \quad -x^2-1 \\ \hline \qquad 1 \end{array}$$

c) • $\displaystyle \int \frac{1}{\sqrt{1-4x}}\,dx = -\frac{1}{4}\;\frac{\sqrt{1-4x}}{\frac{1}{2}} + C = -\frac{1}{2}\sqrt{1-4x} + C \qquad \left(\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\right)$

$x \in \left(-\infty, \frac{1}{4}\right)$

• $\displaystyle \int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\arcsin(2x) + C \qquad \left(\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,\ x \in (-1,1)\right)$

$x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

• $\displaystyle \int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x}{2}\right)\cdot\frac{1}{\frac{1}{2}} + C$

$\displaystyle = \arcsin\left(\frac{x}{2}\right) + C,$

$x \in (-2, 2)$

- $$\int \operatorname{tg}^2 u\,du = \int \frac{\sin^2 u}{\cos^2 u}\,du = \int \frac{1-\cos^2 u}{\cos^2 u}\,du = \int \frac{1}{\cos^2 u}\,du - \int 1\,du =$$
$$= \operatorname{tg} u - u + c\,,\quad u \in \left((2k-1)\tfrac{\pi}{2}, (2k+1)\tfrac{\pi}{2}\right),\ k\in\mathbb{Z}$$

2. Substituce:

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + c\,,\ \text{je-li}\ \int f(y)\,dy = F(y) + c$$
$x \in (\alpha,\beta)$, $g((\alpha,\beta)) \subset (a,b)$ v (a,b)

(předpokládáme $g'(x)$ spojitou v (α,β), f spojitá v (a,b))

---

a) $$\int \frac{1}{\sqrt{x}}\cos\sqrt{x}\,dx = 2\int \cos\sqrt{x}\,(\sqrt{x})'\,dx = 2\sin\sqrt{x} + c\,,\quad x\in(0,+\infty)\quad (*)$$
neboť $\int \cos y\,dy = \sin y + c$

(zde $f(y) = \cos y$, $g(x) = \sqrt{x}$)

Často se „formálně" zapisuje substituce takto (budu dále užívat, ale vy nemusíte, můžete psát jako v (*)):

$$\int \frac{1}{\sqrt{x}}\cos\sqrt{x}\,dx = 2\int \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx = \left|\begin{matrix} \sqrt{x} = y & (\equiv g(x)) \\ \frac{1}{2\sqrt{x}}\,dx = dy & \left(g'(x) = \frac{1}{2\sqrt{x}}\right)\end{matrix}\right| =$$
$$= 2\int \cos y\,dy = 2\sin y + c \underset{(\text{„zpět"})}{=} 2\sin\sqrt{x} + c\,,\quad x\in(0,+\infty)$$

- $\underline{\displaystyle\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx} = -\displaystyle\int \sin\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x}\right)' dx = \cos\left(\frac{1}{x}\right)+c$

  $x\in(-\infty,0) \vee x\in(0,+\infty)$,

  neboť $\int -\sin y\, dy = \cos y + c$

  <u>nebo (jiný zápis)</u>

  $$\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \left| \begin{array}{l} \frac{1}{x}=t \\ -\frac{1}{x^2}dx = dt \end{array} \right| = -\int \cos t\, dt = \sin t + C = \sin\left(\frac{1}{x}\right)+C$$

- $\underline{\displaystyle\int \frac{3x^2}{\sqrt{x^3+8}}dx} = \displaystyle\int \frac{(x^3+8)'}{\sqrt{x^3+8}}dx = 2\sqrt{x^3+8}+c,$

  $x^3+8>0 \Leftrightarrow x\in(-2,+\infty)$

  (opět lze psát: $\displaystyle\int \frac{3x^2}{\sqrt{x^3+8}}dx = \left| \begin{array}{l} x^3+8=t \\ 3x^2dx=dt \end{array} \right| = \int \frac{1}{\sqrt{t}}dt = 2\sqrt{t}+c = 2\sqrt{x^3+8}+c$)

- $\underline{\displaystyle\int \cos^3 x \cdot \sin x\, dx} = -\displaystyle\int \cos^3 x\,(\cos x)' dx = -\frac{\cos^4 x}{4}+c, \quad x\in R$

- $\underline{\displaystyle\int \sin^3 x\, dx} = \displaystyle\int \sin^2 x\cdot\sin x\, dx = \int (1-\cos^2 x)\sin x\, dx =$

  $= \displaystyle\int dx + \int \cos^2 x\,(-\sin x)\,dx = x + \frac{\cos^3 x}{3}+c, \quad x\in R$

b) $\underline{\displaystyle\int \frac{\cos x}{\sin x+3}dx} = \displaystyle\int \frac{(\sin x+3)'}{\sin x+3}dx = \ln|\sin x+3|+c \quad , x\in R$

$(= \ln(\sin x+3)+c)$

zde „užitečný" „vzorec": $\displaystyle\int \frac{g'(x)}{g(x)}dx = \ln|g(x)|+c$ v (a,b), kde $g(x)\neq 0$

(zde $g(x)=\sin x+3$)

$$\underline{\int \frac{1}{1+\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx} = 2\int \frac{1}{1+\sqrt{x}} \cdot \left(\frac{1}{2\sqrt{x}}\right) dx = 2\ln\left(1+\sqrt{x}\right)+C$$

$x \in (0,+\infty)$ (opět $\int \frac{g'(x)}{g(x)}\,dx$, $g(x) = 1+\sqrt{x}$)

$$\underline{\int \operatorname{tg} x\, dx} = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{(+\cos x)'}{\cos x}\, dx = -\ln|\cos x| + C$$

$x \in \left(\left(2k-1\right)\frac{\pi}{2}, \left(2k+1\right)\frac{\pi}{2}\right), k \in \mathbb{Z}$

$$\underline{\int \frac{1}{1+\operatorname{tg} x} \cdot \frac{1}{\cos^2 x}\, dx} = \int \frac{(1+\operatorname{tg} x)'}{1+\operatorname{tg} x}\, dx = \ln|1+\operatorname{tg} x| + C$$

$x \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, $x \in \left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$ atd.

($x \in \left(-\frac{\pi}{2}+k\pi, -\frac{\pi}{4}+k\pi\right)$, $x \in \left(-\frac{\pi}{4}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in \mathbb{Z}$)

c) $$\underline{\int \frac{\ln^2 x}{x}\, dx} = \int \ln^2 x \cdot (\ln x)'\, dx = \frac{\ln^3 x}{3} + C, \quad x \in (0,+\infty)$$

(nebo zápis: $$\int \frac{\ln^2 x}{x}\, dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array} \right| = \int t^2\, dt = \frac{t^3}{3} + C \underset{\text{(opět)}}{=} \frac{\ln^3 x}{3} + C \;)$$

a podobně:

$$\underline{\int \frac{1}{x}\sqrt{1-\ln x}\, dx} = \left| \begin{array}{l} 1-\ln x = t \\ -\frac{1}{x}dx = dt \end{array} \right| = -\int \sqrt{t}\, dt = -\frac{2}{3}t^{3/2} + C = -\frac{2}{3}\left(1-\ln x\right)^{\frac{3}{2}} + C,$$

$1-\ln x > 0 \iff$

$\ln x < 1 \iff x \in (0,e)$

• $$\int \frac{\ln x}{x(1+\ln^2 x)}\,dx = \left| \begin{matrix} \ln x = t \\ \frac{1}{x}dx = dt \end{matrix} \right| = \int \frac{t}{1+t^2}\,dt = \left| \begin{matrix} 1+t^2 = y \\ 2t\,dt = dy \end{matrix} \right|$$

$$= \frac{1}{2}\int \frac{dy}{y} = \frac{1}{2}\ln y + C = \frac{1}{2}\ln(1+\ln^2 x) + C,\ x \in (0,+\infty)$$

nebo, pokud někdo "už vidí" lze substituovat „najednou":

$$\int \frac{\ln x}{x(1+\ln^2 x)}\,dx = \left| \begin{matrix} 1+\ln^2 x = t > 0 \\ 2\ln x \cdot \frac{1}{x}dx = dt \end{matrix} \right| = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln t + C$$

$$= \frac{1}{2}\ln(1+\ln^2 x) + C$$

d) $$\int \frac{e^x}{e^{2x}+2e^x+2}\,dx = \left| \begin{matrix} e^x = t \\ e^x dx = dt \end{matrix} \right| = \int \frac{dt}{t^2+2t+2}\,dt =$$

$$= \int \frac{1}{(t+1)^2+1}\,dt = \operatorname{arctg}(t+1) + C = \operatorname{arctg}(e^x+1) + C,\ x \in R$$

$$\int \frac{\sin x \cdot \cos x}{1+\cos^4 x}\,dx = \left| \begin{matrix} \cos x = t \\ -\sin x\,dx = dt \end{matrix} \right| = -\int \frac{t}{1+t^4}\,dt =$$

$$\left(= -\int \frac{\cos x}{1+\cos^4 x} \cdot (\cos x)'\,dx \right) \qquad \left(= -\frac{1}{2}\int \frac{(t^2)'}{1+(t^2)^2}\,dt \right)$$

"vidíme"

$$\left(\begin{matrix} \text{další} \\ \text{substituce} \end{matrix} \right) = \left| \begin{matrix} t^2 = y \\ 2t\,dt = dy \end{matrix} \right| = -\frac{1}{2}\int \frac{dy}{1+y^2} = -\frac{1}{2}\operatorname{arctg} y + C =$$

$$= -\frac{1}{2}\operatorname{arctg}(\cos^2 x) + C,\quad x \in R$$

Také lze hned "substituovat": $\cos^2 x = t$ $(dt = 2\cos x \cdot (-\sin x))$ pokud "tohle „vidíme"".

$$\int \frac{\sin x \cdot \cos x}{2\sin^2 x + 3\cos^2 x}\,dx = -\frac{1}{2}\int \frac{\left(2\sin^2 x + 3\cos^2 x\right)'}{2\sin^2 x + 3\cos^2 x}\,dx \overset{*}{=}$$

$$\left(2\sin^2 x + 3\cos^2 x\right)' = 4 \cdot \sin x \cdot \cos x - 6\cos x \cdot \sin x = -2\sin x \cdot \cos x$$

$$\overset{*}{=} -\frac{1}{2}\ln\left(2\sin^2 x + 3\cos^2 x\right) + C, \quad x \in R$$

a pokud toto „nevidíme", nic se neděje – $\cos^2 x$ lze „vyjádřit" pomocí $\sin^2 x$ a pak substituovat $\sin x = t$, nebo, obráceně, $\sin^2 x$ lze vyjádřit pomocí $\cos^2 x$ a substituovat $\cos x = t$:

$$\int \frac{\sin x \cdot \cos x}{2\sin^2 x + 3(1-\sin^2 x)}\,dx = \int \frac{\sin x}{3-\sin^2 x}\cdot \cos x\,dx \overset{*}{=} \left|\begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array}\right|$$

$$= \int \frac{t}{3-t^2}\,dt = \left|\begin{array}{l} 3-t^2 = y \\ -2t\,dt = dy \end{array}\right| = -\frac{1}{2}\int \frac{dy}{y} = -\frac{1}{2}\ln|y| + C$$

$$= -\frac{1}{2}\ln\left(3-\sin^2 x\right) + C, \quad x \in R$$

nebo zde „rychleji" substituovat:

$$\overset{*}{=} \left|\begin{array}{l} 3-\sin^2 x = t \\ -2\sin x \cdot \cos x\,dx = dt \end{array}\right| = -\frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln|t| + C$$

$$= -\frac{1}{2}\ln\left(3-\sin^2 x\right) + C$$

analogicky

$$\int \frac{\sin x \cos x}{2\sin^2 x + 3\cos^2 x}\,dx = \int \frac{\cos x \sin x}{2+\cos^2 x}\,dx = -\frac{1}{2}\ln\left(2+\cos^2 x\right) + C$$

(Poznámka: díky tomu, že všechny tři funkce jsou primitivní k téže funkci v R, liší se v R maximálně o konstantu, zde jsou „stejné".)

- $$\int \frac{\cos^3 x}{2+\sin x}\,dx = \int \frac{\cos^2 x}{2+\sin x}\cdot \cos x\,dx = \int \frac{1-\sin^2 x}{2+\sin x}\cdot \cos x\,dx =$$

$$= \left| \begin{matrix} \sin x = t \\ \cos x\,dx = dt \end{matrix} \right| = \int \frac{1-t^2}{2+t}\,dt = -\int \left(t - 2 + \frac{3}{t+2}\right)dt =$$

po vydělení čitatele jmenovatelem

$$= -\left(\frac{t^2}{2} - 2t + 3\ln|t+2|\right) + C = 2\sin x - \frac{\sin^2 x}{2} - 3\ln(\sin x + 2) + C$$

$(t = \sin x$ $\qquad x \in R$

Integrace per partes:

a) • $$\int x^2 \cos x\,dx \underset{pp}{=} \left| \begin{matrix} u' = \cos x, & u = \sin x \\ v = x^2, & v' = 2x \end{matrix} \right| = x^2 \sin x - 2\int x \sin x\,dx =$$

$$\underset{pp}{=} \left| \begin{matrix} u' = \sin x & u = -\cos x \\ v = x, & v' = 1 \end{matrix} \right| = x^2 \sin x - 2\left(-x\cos x + \int \cos x\,dx\right) =$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C, \quad x \in R$$

- $$\int x^3 \ln x\,dx = \left| \begin{matrix} u' = x^3, & u = \frac{x^4}{4} \\ v = \ln x, & v' = \frac{1}{x} \end{matrix} \right| = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx =$$

$$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C, \quad x \in (0, +\infty)$$

- $$\int \ln^2 x\,dx \underset{pp}{=} \left| \begin{matrix} u' = 1 & u = x \\ v = \ln^2 x, & v' = 2\ln x \cdot \frac{1}{x} \end{matrix} \right| = x\ln^2 x - 2\int \ln x\,dx \underset{pp}{=}$$

$$= \left| \begin{matrix} u' = 1, & u = x \\ v = \ln x, & v' = \frac{1}{x} \end{matrix} \right| = x\ln^2 x - 2\left(x\ln x - \int x \cdot \frac{1}{x}\,dx\right) =$$

$$= x\ln^2 x - 2x\ln x + 2x + C, \quad x \in (0, +\infty)$$

b) • $$\underline{\int x \,\mathrm{arctg}\, x \, dx} \underset{PP}{=} \left| \begin{array}{l} u' = x \;, \; u = \frac{x^2}{2} \\ v = \mathrm{arctg}\, x \;, \; v' = \frac{1}{1+x^2} \end{array} \right| =$$

$$= \frac{x^2}{2}\mathrm{arctg}\, x - \frac{1}{2}\int \frac{x^2(+1-1)}{1+x^2}dx = \frac{x^2}{2}\mathrm{arctg}\, x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx =$$

$$= \frac{x^2}{2}\mathrm{arctg}\, x - \frac{1}{2}(x - \mathrm{arctg}\, x) + C = \frac{1}{2}\left[(x^2+1)\mathrm{arctg}\, x - x\right], \; x \in R$$

• $$\underline{\int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}dx}_{\;x\in(0,1)} = \left| \begin{array}{l} u' = \frac{1}{\sqrt{1-x}} \;, \; u = -2\sqrt{1-x} \\ v = \arcsin\sqrt{x} \;, \; v' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \end{array} \right| =$$

$$= -2\sqrt{1-x}\,\arcsin\sqrt{x} + \int \cancel{2}\cancel{\sqrt{1-x}} \cdot \frac{1}{\cancel{\sqrt{1-x}}} \cdot \frac{1}{\cancel{2}\sqrt{x}}dx =$$

$$= -2\sqrt{1-x}\cdot\arcsin\sqrt{x} + 2\sqrt{x} + C \,, \quad x \in (0,1)$$

c) $$\underline{\int \sqrt{1-x^2}\,dx}_{\;x\in(-1,1)} \underset{PP}{=} \left| \begin{array}{l} u' = 1 \;, \; u = x \\ v = \sqrt{1-x^2} \;, \; v' = \frac{1}{\sqrt{1-x^2}}(-x) \end{array} \right| =$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}dx$$

$$= x\sqrt{1-x^2} - \int\sqrt{1-x^2}dx + \int\frac{1}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int\sqrt{1-x^2}dx + \arcsin x + C$$

a odtud (dostali jsme rovnici pro hledaný integrál $\int\sqrt{1-x^2}dx$):

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) + C \,, \quad x \in (-1,1)$$

(řešili jste i substitucí $x = \sin t$)

- $\underline{\int \cos^2 x\,dx} = \left| \begin{matrix} u' = \cos x, & u = \sin x \\ v = \cos x, & v' = -\sin x \end{matrix} \right| = \sin x . \cos x + \int \sin^2 x\,dx =$
  $x \in R$

$= \sin x . \cos x + \int (1 - \cos^2 x)\,dx$, tedy odtud:

$2 \int \cos^2 x\,dx = \sin x \cos x + x + C \Rightarrow \underline{\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C}$
$x \in R$

- $\underline{\int \sin^2 x\,dx} = \int 1\,dx - \int \cos^2 x\,dx = 1 - \frac{1}{2}(x + \sin x \cos x) + C =$
$= \underline{\frac{1}{2}(x - \sin x \cos x) + C}$, $x \in R$

d) • $\underline{\int x^n e^x\,dx} = \left| \begin{matrix} u' = e^x, & u = e^x \\ v = x^n, & v' = n x^{n-1} \end{matrix} \right| = x^n e^x - n \int x^{n-1} . e^x\,dx =$
$x \in R$

$= x^n e^x - n \left(x^{n-1} e^x - (n-1) \int x^{n-2} . e^x\,dx \right) = \dots\dots$

„dále"
$= x^n e^x - n x^{n-1} e^x + n(n-1) x^{n-2} e^x - \dots + (-1)^j n(n-1)\dots(n-j+1) x^{n-j} e^x +$
$\dots + (-1)^n n(n-1)\dots 2.1 . e^x =$

$$= \left(\sum_{j=0}^{n} (-1)^j \frac{n!}{(n-j)!} x^{n-j} \right) e^x \quad , x \in R$$

(možná jednodušší zápis:

$I_n = \int x^n e^x\,dx = x^n e^x - n I_{n-1} = \dots\dots$)